ELSEVIER
Journal of Pure and Applied Algebra 165 (2001) 183–200
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Multiplicative equivariant formal group laws

J.P.C. Greenlees
School of Mathematics and Statistics, Hicks Building, Sheffield, UK S3 7RH,

Received 5 February 1999; received in revised form 31 July 2000
Communicatd by E.M. Friedlander

Abstract

The universal ring for multiplicative equivariant formal group laws is shown to be closely related to the Rees ring of the representation ring at the augmentation ideal, but only equal to it if the group is topologically cyclic. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 14L05; 19L41; 19L47; 55N22; 55N91

1. Introduction

The notion of a A-equivariant formal group law [2] for a compact abelian Lie group A was introduced to study complex oriented A-equivariant cohomology theories (all formal groups in this paper are required to be commutative and one dimensional). The theorem [5] that the coefficient ring of equivariant complex bordism is the universal ring for equivariant formal group laws establishes that the definition is the correct one for this purpose. However, just as a formal group law can be viewed as a parametrized one-dimensional deformation of the trivial group, an equivariant formal group law can also be viewed as a parametrized one-dimensional deformation of a discrete abelian group.

We shall be concerned here with a very special class of equivariant formal group laws: the multiplicative ones, which appear to play a privileged role amongst all equivariant formal group laws. However, our principal motivation for considering this case is its importance in understanding equivariant K-theories, and its close relationship to representation theory. The functorial refinement of (abelian) representation theory seems to be of intrinsic algebraic interest. Much of the algebra presented here is closely mirrored in [1].

E-mail address: j.greenlees@sheffield.ac.uk (J.P.C. Greenlees).

PII: S0022-4049(00)00184-5

Before recalling the definition of an A-equivariant formal group law we need some notation. We let $A^* = \mathrm{Hom}(A, S^1)$ denote the dual group, and ε its neutral element, the trivial character. The letters $\alpha, \beta, \gamma, \dots$ will be used for elements of A^*. We also let k^{A^*} denote the ring of k-valued functions on A^*. If A is finite, this is a Hopf algebra over k using the group multiplication of A^* to give the coproduct, and the inclusion of the identity element in A^* to give the counit.

Definition 1.1 (*Cole et al.* [2]). If A is a finite abelian group, an *A-equivariant formal group law* over a commutative ring k is

(Afgl1) a complete topological Hopf k-algebra R with
(Afgl2) a homomorphism $\theta : R \to k^{A^*}$ of Hopf k-algebras whose kernel defines the topology together with
(Afgl3) an element $y(\varepsilon) \in R$ which is (i) regular and (ii) generates the kernel of the εth component, θ_ε of θ.

Remark 1.2. (i) If A is a general abelian compact Lie group, the definition is the same except that in Condition (Afgl2), R is required to be complete with respect to the system of ideals given by finite intersections of kernels of the components $\theta_\alpha : R \to k$ of θ. The ring k^{A^*} is topologized as the product of discrete rings k, and as such it is a complete topological Hopf algebra. The map θ is required to be a morphism of complete topological Hopf algebras.

(ii) We let $\Delta : R \to R \hat{\otimes} R$ denote the coproduct of the Hopf algebra structure. Since θ is a map of Hopf algebras it follows that θ_ε is the counit of R.

(iii) The element $y(\varepsilon)$ is called the *coordinate* of the formal group law. If the coordinate is not specified, the resulting structure is called an equivariant *formal group*. This terminology arises since by (Afgl1), R may be viewed as the ring of functions on a group object $\mathbb{G}$ in the category of formal schemes over k (see Appendix B for more details). Thus, (Afgl2) states that we are given a homomorphism $\zeta : A^* \to \mathbb{G}$, so that $\mathbb{G}$ is a formal neighbourhood of the image, and (Afgl3) states that $y(\varepsilon)$ is a good coordinate at $\zeta(\varepsilon)$. The group scheme point of view will not be used in the body of the paper, but those familiar with it will find Appendix B an illuminating guide to our formal results.

Definition 1.3. Given an A-equivariant formal group law we may define the *Euler class* of a character α by

$$e(\alpha) = \theta(y(\varepsilon))(\alpha^{-1}).$$

Remark 1.4. (i) One view is that an equivariant formal group law is a structure precisely designed to encode the formal properties of Euler classes.

(ii) When the formal group law arises from a complex-oriented cohomology theory, these coincide with Euler classes in the topological sense [2].

(iii) Note that $e(\varepsilon) = 0$ by (Afgl3)(ii).

The k-module structure of every equivariant formal group law is topologically free, and we may therefore express the structure maps of R in terms of the basis. To describe the basis, we note that we may define an action of A^* on R via $l_\alpha r = (\theta_{\alpha^{-1}} \otimes 1)\Delta(r)$. Thus, the element $y(\varepsilon)$ determines elements $y(\alpha)$ for $\alpha \in A^*$ by the formula $y(\alpha) = l_\alpha y(\varepsilon)$. The completeness is thus equivalent to completeness with respect to the system of principal ideals generated by all finite products $\prod_\alpha y(\alpha)$. In the following statement, a complex complete A-universe is a countably infinite-dimensional complex representation of A in which every simple representation occurs infinitely often.

Theorem 1.5 (Cole et al. [2, 13.2]). *If we choose a complete A-invariant flag $F = (V^1 \subset V^2 \subset \cdots)$ in a complex complete A-universe, then an equivariant formal group law R has an additive topological k-basis* $1, y(V^1), y(V^2), \ldots$ *where* $y(V^n) = y(\alpha_1) y(\alpha_2) \cdots y(\alpha_n)$ *if* $V^n = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$.

Remark 1.6. Note that if A is the trivial group, Theorem 1.5 shows that Definition 1.1 reduces to the usual concept of a (non-equivariant, commutative, one-dimensional) formal group law.

In this note, we consider equivariant formal group laws of a very simple form.

Definition 1.7. (i) An equivariant formal group law R is *multiplicative* if its coproduct has the property

$$\Delta y(\varepsilon) = 1 \otimes y(\varepsilon) + y(\varepsilon) \otimes 1 - v y(\varepsilon) \otimes y(\varepsilon)$$

for some $v \in k$.

(ii) Given a multiplicative formal group law over k, we define a binary operation on k-algebras by $x \odot y = x + y - vxy$.

(iii) We also define a polynomial $[n](x)$ in v and x inductively by $[0](x) = 0$ and $[n](x) = ([n-1](x)) \odot x$. Thus

$$[n](x) = (1 - (1 - vx)^n)/v.$$

Remark 1.8. (i) From Theorem 1.5 the products $y(V^i) \otimes y(V^j)$ give a basis of $R \hat{\otimes} R$ for any flag, so the coproduct determines the element v. The letter v is chosen to correspond to the Bott element in topological K-theory.

(ii) Note that v is not required to be a unit. In particular, we allow the degenerate case $v = 0$, which is usually referred to as an *additive* law. If v is a unit we say the formal group law is *strictly* multiplicative.

(iii) If k is graded, v is homogeneous and x is of degree $-|v|$ then the polynomial $[n](x)$ is also homogeneous and has the same degree as x. In all graded cases we consider, v is of degree 2.

(iv) The notion depends heavily on the coordinate: it is a property of the formal group *law* and not of its underlying formal group. This is somewhat clarified in Appendix B.

The purpose of the present note is to observe that there is a representing ring for multiplicative equivariant formal group laws, to identify it explicitly, and to relate it to representation theory. Readers used to equivariant formal group laws may be surprised by the simplicity of the answer.

The paper is layed out as follows. We begin in Section 2 by explaining how v and the Euler classes determine an A-equivariant formal group law: these calculations also make the definition somewhat more explicit. In Section 3, we make some elementary observations about the Rees ring from commutative algebra. In Section 4, we describe our main results, and the reader may want to glance at it now. In Section 5, we show that our results can be reduced to the cyclic case, and give the proof in that case in Section 6. There are two appendices presenting other ways to think about multiplicative equivariant formal group laws. In Appendix A, we give very explicit formulae for groups of small order. The author recommends Appendix B as particularly illuminating: it discusses the formal group schemes represented by equivariant formal groups. In particular it gives Ando's construction of an A-equivariant formal group law from a fixed one-dimensional group law, and observes that the formal parts of the present paper apply to any class of A-equivariant formal group laws of this type.

2. Euler classes

Suppose a multiplicative A-equivariant formal group law R is given. We explain how to deduce the ring structure on R from v and the Euler classes $e(\alpha)$, and the basic properties of Euler classes. We give the specializations of several proofs from [2] to highlight the simplicity of the argument.

Lemma 2.1. *For any one-dimensional representation* α

$$y(\alpha) = e(\alpha) + (1 - ve(\alpha))y(\varepsilon).$$

Proof. From the definitions, we may calculate

$$\begin{aligned} y(\alpha) &= (\theta_{\alpha^{-1}} \otimes 1)\Delta(y(\varepsilon)) \\ &= (\theta_{\alpha^{-1}} \otimes 1)(1 \otimes y(\varepsilon) + y(\varepsilon) \otimes 1 - vy(\varepsilon) \otimes y(\varepsilon)) \\ &= y(\varepsilon) + e(\alpha) - ve(\alpha)y(\varepsilon) \end{aligned}$$

as required. □

Because the complete universe contains the trivial representation infinitely often, $y(\varepsilon)$ is transcendental over k, and R contains the polynomial ring $k[y(\varepsilon)]$. Now the very special feature of the multiplicative group is that by Lemma 2.1, the elements $y(\alpha)$ all lie in this polynomial ring. Accordingly, the ring R is much simpler than for a general equivariant formal group law, and can be expressed entirely in terms of v and the Euler classes.

Corollary 2.2. *The coordinate $y(\varepsilon)$ is a topological generator of R, and R is the completion of the polynomial ring $k[y(\varepsilon)]$ for the topology defined by all finite products $y(\alpha_1)y(\alpha_2)\cdots y(\alpha_n)$.*

Lemma 2.3. *The Euler classes determine the structure map θ.*

Proof. Since θ is a continuous ring homomorphism it suffices to identify $\theta(y(\beta))$ for all $\beta \in A^*$. Indeed, we calculate

$$\theta(y(\beta))(\alpha^{-1}) = \theta(y(\varepsilon))(\alpha^{-1}\beta^{-1}) = e(\alpha\beta),$$

where the first equality follows by applying l_β, since θ is a Hopf map. □

Lemma 2.4. *The element v and the Euler classes determine the coproduct Δ.*

Proof. Since Δ is a continuous ring homomorphism it suffices to calculate its value on $y(\alpha)$. For this we have

$$\Delta(y(\alpha)) = (\Delta \circ l_\alpha)(y(\varepsilon)) = ((1 \otimes l_\alpha) \circ \Delta)(y(\varepsilon)).$$

Since we calculated $l_\alpha y(\varepsilon)$ in terms of v and the Euler classes in Lemma 2.1, this gives the required formula. □

This completes the explanation of how v and the Euler classes determine the structure of a multiplicative A-equivariant formal group. In fact, we may go a little further since the coproduct describes the Euler classes of tensor products, and so the Euler classes $e(\alpha)$ for a set of generators α of the abelian group A^* determine all Euler classes.

First, we define the polynomial $[n](x)$ inductively by $[0](x) = 0$ and $[n](x) = ([n-1](x)) \odot x$. Thus,

$$[n](x) = (1 - (1 - vx)^n)/v.$$

Lemma 2.5. *The Euler class of a tensor product is described by the formula*

$$e(\alpha\beta) = e(\alpha) \odot e(\beta).$$

Furthermore,

$$e(\alpha^n) = [n](e(\alpha))$$

and

$$e(\alpha^n) = e(\alpha^{n-1}) + e(\alpha)(1 - ve(\alpha))^{n-1}.$$

Proof. The first formula follows from the fact that the structure map θ is a map of Hopf algebras. The resulting equation on $y(\varepsilon)$ gives the formula when evaluated at $(\alpha^{-1}, \beta^{-1})$. The remaining formulae are immediate consequences. □

Note that if $\alpha^n = \varepsilon$ then we have $[n](e(\alpha)) = 0$. This is slightly stronger than the statement that $(1 - ve(\alpha))^n = 1$.

Corollary 2.6. *For any one-dimensional representation* α, *the element* $1 - ve(\alpha)$ *is a unit with inverse* $1 - ve(\alpha^{-1})$.

Proof. We have

$$(1 - ve(\alpha))(1 - ve(\beta)) = 1 - ve(\alpha) \odot e(\beta) = 1 - ve(\alpha\beta),$$

so that $1 - ve(\alpha^{-1})$ is inverse to $1 - ve(\alpha)$. □

3. The Rees ring of the representation ring

To prepare for the statements of our main results, we recall a construction from commutative algebra. For any commutative ring R, we may form the $\mathbb{Z}$-graded ring $R[v, v^{-1}]$ in which v is assigned degree 2. The *Rees ring* $\mathrm{Rees}(R, J)$ associated to the ideal J of R is the graded subring of $R[v, v^{-1}]$ whose homogeneous parts are powers of J:

$$\mathrm{Rees}(R,J)_{2n} = \begin{cases} Rv^n & \text{if } n \geq 0, \\ J^{-n}v^n & \text{if } n \leq 0. \end{cases}$$

Now consider complex representation ring $R(A)$. For any complex representation V, we may define the Euler classes $\chi(V)$ as the alternating sum of exterior powers of V. Thus, if α is one-dimensional, $\chi(\alpha) = 1 - \alpha$, and $\chi(V \oplus W) = \chi(V)\chi(W)$. Note that since A is abelian, the augmentation ideal $J = \ker(R(A) \to \mathbb{Z})$ is generated by the Euler classes $\chi(\alpha)$ of one-dimensional representations, and more generally J^n is generated by the Euler classes of n-dimensional representations. It will be useful to have economical sets of generators for J and the Rees ring.

Lemma 3.1. *The ideal* J *is generated by the Euler classes* $\chi(\alpha)$ *as* α *runs through a set of generators for the dual group* A^*.

Proof. The formula $1 - \alpha\beta = (1 - \alpha) + \alpha(1 - \beta)$ allows us to obtain $\chi(\alpha\beta)$ from $\chi(\alpha)$ and $\chi(\beta)$. □

Lemma 3.2. *The Rees ring* $\mathrm{Rees}(R(A), J)$ *is generated as a ring by* v *and the shifted Euler classes* $e(V) = v^{-|V|}\chi(V)$ *of representations, where* $|V|$ *denotes the complex dimension of the representation* V. □

Lemma 3.3. *The shifted Euler classes satisfy the relation*

$$e(\alpha\beta) = e(\alpha) \odot e(\beta). \quad \square$$

Remark 3.4. It will transpire that there is an A-equivariant formal group law over $k = R(A)[v, v^{-1}]$ whose Euler classes are the elements $e(\alpha) = (1 - \alpha)/v$. The notation for shifted Euler classes is therefore reasonable. This law is multiplicative, and topologists will recognize k as the coefficient ring of equivariant K-theory $K_A^* = R(A)[v, v^{-1}]$.

It is convenient to record here the following elementary fact.

Lemma 3.5. *If x is a list of polynomial variables and $f \in \mathbb{Z}[v,x]$ is not a multiple of v then $\mathbb{Z}[v,x]/(f)$ has no v-torsion.*

Proof. If $t \in \mathbb{Z}[v,x]$ represents a v-torsion element of $\mathbb{Z}[v,x]/(f)$ then $vt = fg$ for some $g \in \mathbb{Z}[v,x]$. Since v is prime and $f \neq 0 \bmod v$ it follows that $g = vh$ for some $h \in \mathbb{Z}[v,x]$. Since $\mathbb{Z}[v,x]$ has no v-torsion we deduce $t = fh$, so t represents 0. □

4. Universal rings for multiplicative formal group laws

In this section, we describe the universal multiplicative and additive equivariant formal group laws. First, we note that the set A-fgl(k) of A-equivariant formal group laws over k is a functor of the ring k. Indeed, if $f : k \to l$ is a ring homomorphism and R is an A-equivariant formal group law over k then we may define an A-equivariant formal group law f_*R over l by applying $\hat{\otimes}\, l$. The result is again an A-equivariant formal group, since by 1.5 R is a topologically free k-module. In other words, we use the fact that the structure of R may be described by certain structure constants in k, and let f_*R be described by their images in l. It is shown in [2] that the functor A-fgl$(\cdot)$ is represented by a ring L_A in the sense that

$$A\text{-fgl}(k) = \mathrm{Ring}(L_A, k),$$

although we do not need to use this fact here. Given its existence, L_A may be constructed by giving generators for each of the structure constants, and imposing relations to ensure that the axioms of Definition 1.1 hold. The A-equivariant formal group law over k corresponding to a ring homomorphism $f : L_A \to k$ is the one with structure constants given by the image of the corresponding generators of L_A.

Since multiplicative formal group laws are defined by the vanishing of most terms in the coproduct, the set A-fgl$_m(k)$ of multiplicative equivariant formal group laws is also a functor of k. It follows from the existence of L_A that there is also a representing ring L_A^m for multiplicative A-equivariant formal group law functor:

$$A\text{-fgl}_m(k) = \mathrm{Ring}(L_A^m, k),$$

although we shall actually prove the existence of L_A^m by constructing it. Similarly, there are representing rings L_A^a and L_A^{sm} for additive and strictly multiplicative A-equivariant formal group laws. The results in this section give explicit presentations of these representing rings.

In Section 2, we showed that the entire structure of a multiplicative A-equivariant formal group law over k is determined (independently of the ring k) by polynomials in the element v and the Euler classes $e(\alpha)$ of a set of characters α generating A^*: this shows that L_A^m is generated by elements v and $e(\alpha)$ corresponding to these structure constants. The fact that L_A^m is universal implies that the identity map of L_A^m defines an A-equivariant multiplicative formal group law over L_A^m itself; this is *the universal A-equivariant multiplicative formal group law*. This legitimizes the names of the generators, since v is the coefficient in the coproduct $\Delta_{univ}(y(\varepsilon))$ in Definition 1.7, and $e(\alpha)$ is the Euler class of α for this universal law.

The results of Section 2 show how to construct any multiplicative A-equivariant formal group law (its ring, its coproduct and its map θ) from its structure constants, and applies in particular to give the universal law from the following descriptions of the representing rings.

The main content in the descriptions of the universal rings is in showing that all relations follow from relations on the Euler classes that we have already met in Section 2.

Theorem 4.1. *For any compact abelian Lie group A there is a representing ring L^m_A for multiplicative equivariant formal group laws. The ring L^m_A is a $\mathbb{Z}[v]$-algebra and may be described as follows*:

(i) *If $A = B \times C$ then*

$$L^m_A = L^m_B \otimes_{\mathbb{Z}[v]} L^m_C.$$

(ii) *If A is a finite cyclic group of order n with dual group $A^* = \langle \alpha \rangle$ then*

$$L^m_A = \mathbb{Z}[v, e]/([n](e)),$$

where $e = e(\alpha)$. This becomes a graded ring if v has degree 2 *and e is of degree* -2.

(iii) *If A is a circle group and $A^* = \langle z \rangle$ then*

$$L^m_A = \mathbb{Z}[v, f, f']/(vff' - f - f'),$$

where $f = e(z)$ and $f' = e(z^{-1})$. This becomes a graded ring if v has degree 2 *and both f and f' have degree* -2.

We may make this more explicit by choosing presentations. Suppose $A = B \times C$ where B is finite and C is a d-dimensional torus, so that we have the presentation

$$A^* = \langle \beta_1, \beta_2, \ldots, \beta_r, z_1, z_2, \ldots, z_d \mid \beta_1^{n_1}, \beta_2^{n_2}, \ldots, \beta_r^{n_r} \rangle$$

of the dual group for suitable integers $n_1, n_2, \ldots, n_r \geq 2$. We write $e_i = e(\beta_i)$, and $f_j = e(z_j)$ and $f'_j = e(z_j^{-1})$.

Corollary 4.2. *With the above notation*

$$L^m_A = \mathbb{Z}[v, e_1, e_2, \ldots, e_r, f_1, f'_1, f_2, f'_2, \ldots, f_d, f'_d]/\mathfrak{a}$$

where the ideal of relations is

$$\begin{aligned}\mathfrak{a} \;=\; &([n_1](e_1), [n_2](e_2), \ldots, [n_r](e_r),\\ &vf_1f'_1 = f_1 + f'_1, vf_2f'_2 = f_2 + f'_2, \ldots, vf_df'_d = f_d + f'_d).\end{aligned}$$

Remark 4.3. To clarify the logic of the rest of the paper, let

$$\tilde{L}^m_A := \mathbb{Z}[v, e_1, e_2, \ldots, e_r, f_1, f'_1, f_2, f'_2, \ldots, f_d, f'_d]/\mathfrak{a}\,.$$

The rest of this section is devoted to giving elementary proofs of properties of the commutative ring $\tilde{L}^m_A$. We will use Theorem 4.1 *only* to state the conclusions as properties of L^m_A.

Corollary 4.4. *There is a natural map*

$$\nu : L^m_A \to R(A)[v, v^{-1}],$$

with image equal to the Rees ring.

Proof. We define the map $\tilde{\nu} : \tilde{L}^m_A \to R(A)[v, v^{-1}]$ by $\tilde{\nu}(v) = v$ and $\tilde{\nu}(e(\alpha)) = e(\alpha)$, where $e(\alpha)$ on the right is the shifted Euler class $(1 - \alpha)/v$. This is legitimate, since by 3.3, the defining relations also hold in $R(A)[v, v^{-1}]$.

Since Lemmas 3.1 and 3.2 show that the Rees ring is generated by v together with e's, f's and f''s, the image is as claimed. □

The above description of L^m_A depends strongly on the chosen presentation of the group A. If A is topologically cyclic we have a more satisfactory description.

Proposition 4.5. (i) *If A is topologically cyclic then the map ν of Corollary* 4.4 *induces an isomorphism*

$$L^m_A \cong \mathrm{Rees}(R(A), J).$$

(ii) *For any abelian group A, the map ν is the localization away from v*:

$$L^m_A[v^{-1}] \cong R(A)[v, v^{-1}].$$

(iii) *If A is not topologically cyclic then L^m_A contains $\mathbb{Z}$-torsion and v-torsion.*

Remark 4.6. According to Remark 4.3, the corresponding conclusion with L^m_A replaced by $\tilde{L}^m_A$ holds independently of 4.1, and will be used to prove the cyclic case of Theorem 4.1.

Proof. In view of Part (iii), it is appropriate to give the proofs of Parts (i) and (ii) in some detail.

We begin by proving Part (ii). If A is cyclic of order n, the relation $[n](e) = 0$ is equivalent to $(1 - ve)^n = 1$ once v is inverted, so Part (ii) follows the presentation $R(A) = \mathbb{Z}[\alpha]/(\alpha^n = 1)$. If A is the circle group, the relation $vff' = f + f'$ becomes equivalent to $(1 - vf)(1 - vf') = 1$ once v is inverted, so Part (ii) follows from the presentation $R(A) = \mathbb{Z}[z, z']/(zz' = 1)$. Part (ii) now follows in general, since both the functors $\tilde{L}^m_A[1/v]$ and $R(A)[v, v^{-1}]$ take products of abelian groups to tensor products over $\mathbb{Z}[v]$.

In view of Corollary 4.4, the only thing to be proved for Part (i) is that ν is injective when A is topologically cyclic. By Part (ii), it is equivalent to check that $\tilde{L}^m_A$ has no v-torsion in this case. When A is a circle or a finite cyclic group, $\tilde{L}^m_A$ has no v-torsion by Lemma 3.5, since the polynomials $[n](x)$ and $vxy - x - y$ are not multiples of v. We argue by induction on the dimension of A, and the result is true if A is 0-dimensional. Now, if A is topologically cyclic and of dimension ≥ 1, it is of the form $B \times C$ with B of lower dimension and C a circle. Observe that $v_{B\times C} = v_B \otimes_{\mathbb{Z}[v]} \tilde{L}^m_C$. Since v_B is injective by induction, the injectivity of v_A follows from the less obvious fact that $\mathbb{Z}[v,f,f']/(vff' - f - f')$ is flat over $\mathbb{Z}[v]$ [6, 22.6].

Now, if A is not topologically cyclic there are independent elements $\alpha, \beta \in A^*$ of order p for some prime p. Furthermore, we may suppose they lie in a subgroup B^* which is a retract of A^* and of form $B^* = C_{p^a} \times C_{p^b}$ for some $a, b \geq 1$. Thus $A = B \times C$, and so $L^m_A = L^m_B \otimes_{\mathbb{Z}[v]} L^m_C$ by Theorem 4.1(i); since L^m_A is augmented over $\mathbb{Z}[v]$ it follows that L^m_B is a $\mathbb{Z}[v]$-subalgebra of L^m_A, and we may thus suppose $A^* = C_{p^a} \times C_{p^b}$.

Let $e = e(\alpha)$ and $f = e(\beta)$. We thus have $pe = ve^2 s(e)$ and $pf = vf^2 s(f)$ for a polynomial $s(x) \in \mathbb{Z}[v][x]$ of degree $p-2$. Hence $t = fe^2 s(e) - ef^2 s(f)$ is v-torsion and therefore et and ft are p-torsion. To see that t, et and ft are themselves non-zero it suffices to check this mod p. Working mod p we find $[p](x) = -v^{p-1}x^p$, so the relevant ring is

$$L^m_A/p = \mathbb{Z}/p[v,x,y]/((vx)^{p^a}/v, (vy)^{p^b}/v)$$

with $e = (vx)^{p^{a-1}}/v$ and $f = (vy)^{p^{b-1}}/v$, and $t = v^{p-2}(ef^p - e^p f)$. $\square$

Corollary 4.7. *If A is topologically cyclic, the representing ring L^m_A for multiplicative A-equivariant formal group laws, may be identified with the Rees ring*

$$L^m_A = \mathrm{Rees}(R(A), J).$$

In any case, the representing ring for strictly multiplicative A-equivariant formal group laws is given by

$$L^{sm}_A = R(A)[v, v^{-1}].$$

Remark 4.8. (1) If we set $v = 1$, we recover the observation of [2] that the universal ring for equivariant multiplicative formal group laws of the form

$$\Delta y(\varepsilon) = 1 \otimes y(\varepsilon) + y(\varepsilon) \otimes 1 - y(\varepsilon) \otimes y(\varepsilon)$$

is the representation ring $R(A)$.

(2) The ring $L^{sm}_A = L^m_A[v^{-1}] = R(A)[v, v^{-1}]$ is the coefficient ring of equivariant periodic complex K-theory.

(3) It is shown in [3] that if A is of prime order, there is a good equivariant form of *connective* complex K-theory *ku*, and that its coefficient ring is L^m_A. In fact, this is also true when A is any product of two topologically cyclic groups [4]. However, it cannot be true for all abelian groups. Indeed, the completion of the coefficient ring of A-equivariant connective K-theory must be $ku^*(BA)$, and this usually has non-zero groups in odd degrees (for example if A is elementary abelian of rank ≥ 3). It would be very interesting to have a purely algebraic prediction for the coefficient ring of equivariant connective K-theory in general.

Finally, we record the corresponding results for additive formal group laws, which follow by setting $v=0$.

Corollary 4.9. *There is a universal ring L_A^a for additive A-equivariant formal group laws. It is the free commutative ring on the abelian group A^*, and with the above notation for A^*, it has the presentation*

$$L_A^a=\mathbb{Z}[e_1,e_2,\ldots,e_r,f_1,f_2,\ldots,f_d]/(n_1e_1,n_2e_2,\ldots,n_re_r). \qquad \square$$

5. Decoupling and its consequences

The purpose of this section is to show that for multiplicative formal group laws the coproduct and Euler classes can be largely separated. This then allows us to give the formal reduction of the main theorem to the cases of the finite cyclic groups and the circle.

An equivariant formal group is a more complicated object than a non-equivariant one. In the non-equivariant case, a coordinate gives an isomorphism $R=k[[y]]$ and the coproduct is defined relative to that ring structure. However, in general the ring structure on R depends on the structure map θ, and the formulation of the condition that θ is a Hopf map requires recursive use of θ itself. Fortunately, things are simpler in the multiplicative case. First note that the multiplicative coproduct Δ is only polynomial and restricts to a coproduct on $k[y]$. This makes $k[y]$ into a Hopf algebra (but note that it does not admit an antipode (see Appendix B for further discussion)). We may now prove the key result separating the two parts of the structure for multiplicative group laws.

Proposition 5.1 (Decoupling of coproduct and Euler classes). *Any map $\theta':k[y]\to k^{A^*}$ of topological Hopf algebras from a multiplicative Hopf algebra determines a unique A-equivariant multiplicative formal group law whose structure map θ extends θ'.*

Proof. First note that we may define Euler classes by $e(\alpha)=\theta'(y)(\alpha^{-1})$, and $e(\varepsilon)=0$ since it is the augmentation of y. We may now define a topology on $k[y]$ by taking $y(\alpha)=e(\alpha)+(1-ve(\alpha))y$ in line with Lemma 2.1. Next, we claim that θ' is continuous for the topology. For this it suffices to note that

$$\begin{aligned}\theta'(y(\alpha))(\beta) &= \theta'(e(\alpha)+(1-ve(\alpha))y)(\beta)\\ &= e(\alpha)+(1-ve(\alpha))e(\beta^{-1})\\ &= e(\alpha)\odot e(\beta^{-1})\\ &= e(\alpha\beta^{-1}),\end{aligned}$$

so that $\theta'(y(\alpha))$ vanishes in the αth coordinate since $e(\varepsilon)=0$.

We may now let R be the completion of $k[y]$ for this topology. It is clear that the multiplicative coproduct extends to R, and continuity of θ' ensures that it extends to a map θ. Finally we take $y(\varepsilon):=y$ and verify Conditions (Afgl3)(i) and (ii) of Definition 1.1. For (i), suppose that $y\mathbf{f}=0$ for some $\mathbf{f}\in R$ represented by the sequence $(f_n(y))_n$ of polynomials. Thus, the sequence $(yf_n(y))_n$ tends to zero, so that any finite product of $y(\alpha)$'s divides some $yf_n(y)$. Since y is regular on $k[y]$ it follows that the sequence $(f_n(y))_n$ also tends to zero and $\mathbf{f}=0$ as required. For (ii), it is immediate that y lies in $\ker(\theta_\varepsilon)$. On the other hand, suppose $\theta_\varepsilon(\mathbf{f})=0$ for some $\mathbf{f}\in R$ represented by the sequence $(f_n(y))_n$ of polynomials. Note that since $f_n(y)$ is a convergent sequence and k is discrete, $\theta(f_n(y))(\varepsilon)$ is ultimately constant. Since $\theta(\mathbf{f})=0$, the constant value is zero, so that $\theta(f_n(y))(\varepsilon)=f_n(0)$ provided n is sufficiently large. Thus y divides $\mathbf{f}$ as required.

Finally, uniqueness of the formal group law follows since $k[y]$ is always dense by Corollary 2.2. □

Corollary 5.2. *A multiplicative A-equivariant formal group law over k is given by an element $v\in k$ and a map $\theta' : k[y]\to k^{A^*}$ of topological Hopf algebras.*

We may now easily explain how the proof of the main theorem may be reduced to the special cases when A is the circle or a finite cyclic group. Note first that an arbitrary abelian compact Lie group is a product of these special groups: this product decomposition propogates through the entire structure.

For the following two well-known lemmas, think of Hopf algebras as group objects in the category of cocommutative coalgebras (so in particular they are cocommutative).

Lemma 5.3. *If H_1 and H_2 are complete topological Hopf algebras then $H_1\hat{\otimes} H_2$ is also a complete topological Hopf algebra, and it is the categorical product.*

Proof. It is a formality that the forgetful map from group objects in a category to all objects creates products. It therefore suffices to check that the completed tensor product of two complete topological coalgebras is their categorical product. □

Lemma 5.4. *For discrete abelian groups B', C' there is a natural isomorphism*

$$k^{B'\times C'} \cong k^{B'} \hat{\otimes} k^{C'}$$

expressing $k^{B'\times C'}$ as a categorical product of Hopf algebras using the group projections.

The proof of Part (i) of Theorem 4.1 is now a formality.

Corollary 5.5. *If $A=B\times C$ then $L^m_A=L^m_B\otimes_{\mathbb{Z}[v]} L^m_C$.*

Proof. We saw in Corollary 5.2 that an equivariant formal group law is specified by $v\in k$ together with a topological Hopf map $\theta : k[y]\to k^{A^*}$.

Fix v, and note that since $k^{(B\times C)^*}$ is the product of k^{B^*} and k^{C^*} as topological Hopf algebras,

$$\mathrm{Hopf}(k[y],k^{A^*})=\mathrm{Hopf}(k[y],k^{B^*})\times \mathrm{Hopf}(k[y],k^{C^*}).$$

It follows that the representing ring is the coproduct of L^m_B and L^m_C. $\square$

6. Proof of the main theorem

After Section 5, it remains only to prove Theorem 4.1 Parts (ii) and (iii). As before, we let $\tilde{L}^m_A=\mathbb{Z}[v,e]/[n](e)$ if A is cyclic of order n or $\mathbb{Z}[v,f,f']/vff'=f+f'$ if A is the circle. Since the specified relation holds in all multiplicative formal group laws by Lemma 2.5, we have a natural map $\tilde{L}^m_A\to L^m_A$, and we must show it is an isomorphism.

It was proved in Section 2 that if R is a multiplicative equivariant formal group law, then its structure is determined by the elements v and e if A is finite or v,f and f' if A is the circle. This shows the map is a surjective. To complete the proof it suffices to show that there is an A-equivariant formal group law over $\tilde{L}^m_A$ for which the structure constants are as implied by the nomenclature of the generators of $\tilde{L}^m_A$. By 4.6 $\tilde{L}^m_A$ is a subring of $\tilde{L}^m_A[v^{-1}]=R(A)[v,v^{-1}]$, and it suffices to show there is such an A-equivariant formal group law over $k=R(A)[v,v^{-1}]$. Since v is not a zero divisor in k, we may specify 'Euler classes' by $ve(\alpha)=1-\alpha$, and it remains only to check that the corresponding map θ is a map of Hopf algebras.

By Proposition 5.1, it suffices to consider the restriction $\theta':k[y]\to k^{A^*}$, defined by $\theta'(y)(\alpha)=e(\alpha^{-1})$. The fact that the resulting map θ' is a map of Hopf algebras may be verified by evaluation on y, and this is the calculation

$$\begin{aligned} ve(\alpha\beta) &= 1-\alpha\beta=(1-\alpha)+(1-\beta)-(1-\alpha)(1-\beta)\\ &= ve(\alpha)+ve(\beta)-ve(\alpha)ve(\beta). \qquad \square\end{aligned}$$

Remark 6.1. Topologists will note that the existence of the appropriate equivariant formal group law over $R(A)[v,v^{-1}]$ also follows from the fact that equivariant K-theory together with the Euler class of the canonical line bundle is a complex oriented theory. However, this relies on equivariant Bott periodicity, and is therefore much less elementary.

Acknowledgements

The author is grateful to M. Ando and R.R. Bruner for useful discussions and comments, and to the referee for his critical comments which led to the present substantial revision of the first submission.

Appendix A. The structure of multiplicative equivariant formal group laws

Note that Corollary 2.2 shows that, for any A, the underlying ring R of an equivariant formal group law can be described as a completion of the polynomial ring $k[y]$ at the finite products $\prod_\alpha y(\alpha)$. Collecting together the results of Section 2, we are able to give a more explicit description when A is finite. This makes the geometry of the situation a little clearer.

For the rest of the section, we assume A is finite and adopt the abbreviations $y = y(\varepsilon)$ and $x = \Pi_\alpha y(\alpha)$.

Proposition A.1. *If A is a group of order N, then*

$$R = k[[x]][y]/(y^N = ux + yr(y))$$

for some polynomial $r(y)$ of degree $\leq N-2$ and some unit u.

Remark A.2. The proof will show that the polynomial $r(y)$ and the unit u are essentially independent of R. More precisely, the element u and the coefficients of $r(y)$ can be expressed as elements of $\mathbb{Z}[v, e_1, e_2, \ldots, e_r]$ where $e_i = e(\beta_i)$ is the Euler class of the ith generator β_i of A^*.

Indeed, the proof will give an algorithm for finding u and $r(y)$ explicitly. For instance, if A is cyclic we may choose a generator α of A^* and take $e = e(\alpha)$ to obtain

$$R = k[[x]][y]/(y^2 = (1 - ev)x + ey) \quad \text{if } A \text{ is of order } 2$$

and

$$R = k[[x]][y]/(y^3 = x + ey(y(3 - ev) - e(2 - ev))) \quad \text{if } A \text{ is of order } 3.$$

Proof. Certainly there is a natural map $k[x, y] \to R$, determined by our choice of coordinate, and this extends to the completion $k[[x]][y]$. The map is surjective by Corollary 2.2.

Now, choose a periodic complete flag with $\alpha_1 = \varepsilon$, $\alpha_i = \alpha_{i+N}$ for all i and $V^{kN} = k\mathbb{C}A$. By Theorem 1.5, we know $1, y = y(V^1), y(V^2), \ldots, x = y(V^N), xy = y(V^{N+1}), xy(V^2) = y(V^{N+2}), \ldots, x^2 = y(V^{2N}), x^2y = y(V^{2N+1}), \ldots$ are topologically independent over k. It therefore suffices to establish the relation $y^N = ux + yr(y)$ in R. This is a special case of the next lemma, which applies whether A is finite or not. $\square$

Lemma A.3. *For any $n \geq 1$ there is a relation $y^n = u_n y(\alpha_1)y(\alpha_2)\cdots y(\alpha_n) + yr_n(y)$ in R where u_n is a unit and $r_n(y)$ is of degree $\leq n-2$. The element u_n and the coefficients of $r_n(y)$ can be expressed as elements of $\mathbb{Z}[v, e_1, e_2, \ldots, e_s]$ where the elements $e_1, e_2, \ldots, e_s$ are Euler classes of monoid generators of A^*. We have the recursive formulae $u_1 = 1$, $r_1(y) = 0$ and for $n \geq 2$,*

$$u_{n+1} = u_n(1 - ve(\alpha_{n+1}^{-1}))$$

and

$$r_{n+1}(y) = r_n(y)[y + e(\alpha_{n+1})(1 - ve(\alpha_{n+1}^{-1}))] - y^{n-1}e(\alpha_{n+1})(1 - ve(\alpha_{n+1}^{-1})).$$

Proof. We prove this by induction on n, noting it is trivial for $n = 1$. For the inductive step, we suppose the result is true as stated and note that $y(\alpha_{n+1}) = e(\alpha_{n+1}) + (1 - ve(\alpha_{n+1}))y$ by Lemma 2.1. Since $(1 - ve(\alpha_{n+1}))$ is a unit by Corollary 2.6

we obtain

$$y^{n+1} = u_n y(\alpha_1)y(\alpha_2)\cdots y(\alpha_n)(1 - ve(\alpha_{n+1}))^{-1}(y(\alpha_{n+1}) - e(\alpha_{n+1})) + y^2 r_n(y).$$

Noting that any Euler class can be expressed as a polynomial in $e_1, e_2, \ldots, e_r$ by 2.5, this gives an equation of the required form. □

Appendix B. The represented group schemes

In this section, we make explicit how the topological k-algebra R represents a formal group $\hat{\mathbb{G}}$. By definition, $\hat{\mathbb{G}}$ is the set valued functor on k-algebras whose l-valued points are the continuous k-algebra homomorphisms into l:

$$\hat{\mathbb{G}}(l) = k\text{-Alg}_{cts}(R, l).$$

The coproduct on R gives $\hat{\mathbb{G}}(l)$ the structure of an abelian group. Furthermore, the homomorphism θ defines a group homomorphism

$$\zeta : A^* \to \hat{\mathbb{G}}(l)$$

by the formula $\zeta(\alpha)(r) = \theta_\alpha(r)$.

Lemma B.1. *Evaluation at $y(\varepsilon)$ gives an identification*

$$\hat{\mathbb{G}}(l) = A\text{-nil}(l)$$

where A-nil(l) is the ideal of l defined by

$$A\text{-nil}(l) = \{x \in l \mid \textstyle\prod_\alpha (e(\alpha) - (1 - ve(\alpha))x) \textit{ is topologically nilpotent}\}.$$

Under this identification, the group operation is given by

$$x \odot y = x + y - vxy.$$

If A is infinite, the statement about topological nilpotence is to be interpreted as stating that a sequence of products of elements $(e(\alpha) - (1 - ve(\alpha))x)$ tends to zero provided each representation α occurs infinitely often.

Proof. Because $y(\varepsilon)$ generates R, a map $R \to l$ is determined by its image, and we may view $\hat{\mathbb{G}}(l)$ as a subset of l. The given description of A-nil(l) follows from Corollary 2.2. □

This shows that $\hat{\mathbb{G}}(l)$ can be viewed as a subset of l as in the classical situation. However, there are two differences from a classical formal group law. Firstly, the element $y(\varepsilon)$ is not usually topologically nilpotent, and secondly, it is not a free generator. From the geometric point of view, we may ask how to think of a point $x \in \hat{\mathbb{G}}(l)$. Classically, only topologically nilpotent elements of l qualify: these are point infinitesimally close to the identity $0 = \zeta(\varepsilon)$. In the equivariant case, a point of l infinitesimally close to any of the points $\zeta(\alpha)$ in the image of ζ qualifies. Thus, $\hat{\mathbb{G}}(l)$ is an infinitesimally thickened copy of A^* in l.

It is also instructive to describe the proof of the main result using the language of group schemes. First, recall the multiplicative group scheme $\mathbb{G}_m$, defined by

$$\mathbb{G}_m(l) = k\text{-Alg}(k[z, z^{-1}], l) = \text{Units}(l),$$

where the group multiplication is induced by the coproduct $\Delta(z) = z \otimes z$ and the inverse by $\iota(z) = z^{-1}$. The universal example is the case with $k = \mathbb{Z}$. Now take $x = 1 - z$ as a coordinate and note that the coproduct takes the form $\Delta(x) = x \otimes 1 + 1 \otimes x - x \otimes x$. If v is a unit, we may replace x by $y = v^{-1}x$ and obtain the coproduct $\Delta(y) = y \otimes 1 + 1 \otimes y - vy \otimes y$ in the form we have been discussing. But note that to define the inverse it is essential that $z = 1 - vy$ is invertible. Of course, this holds for the rings $k[y, (1 - vy)^{-1}] = k[z, z^{-1}]$ and $k[[y]]$, which define the multiplicative group $\mathbb{G}_m$ and the multiplicative formal group $\hat{\mathbb{G}}_m$ over k, at least provided v is invertible in k.

By contrast, we need to discuss the multiplicative monoid scheme $\mathbb{M}_{m,v}$ with parameter v. This is defined by

$$\mathbb{M}_{m,v}(l) = k\text{-Alg}(k[y], l)$$

with monoid structure defined by the coproduct $\Delta(y) = y \otimes 1 + 1 \otimes y - vy \otimes y$. We choose y as a coordinate at the identity to obtain a monoid law. By construction, there is a canonical homomorphism

$$\hat{\mathbb{G}} \to \mathbb{M}_{m,v}$$

of monoid schemes determined by the coordinate $y(\varepsilon)$. Furthermore, it is shown in Section 2 that any monoid homomorphism $\zeta' : A^* \to \mathbb{M}_{m,v}$ factors uniquely through a group homomorphism $\zeta : A^* \to \hat{\mathbb{G}}$. Thus, A-equivariant formal group laws with parameter v correspond to homomorphisms ζ', and

$$A\text{-fgl}_{m,v}(l) = \text{Monoid}(A^*, \mathbb{M}_{m,v}(l)).$$

The universal case has $k = \mathbb{Z}[v]$, and we conclude

$$A\text{-fgl}_m = \text{Monoid}(A^*, \mathbb{M}_m)$$

where $\mathbb{M}_m$ is the universal multiplicative non-equivariant formal group law $\mathbb{M}_{m,v}$ over $\mathbb{Z}[v]$. Thus, the representing ring L_A^m is the ring of functions on $\text{Monoid}(A^*, \mathbb{M}_m)$.

To calculate the ring of functions we observe (using Section 5 for the detailed justification) that if $A = B \times C$ then

$$\text{Monoid}(A^*, \mathbb{M}_{m,v}) = \text{Monoid}(B^*, \mathbb{M}_{m,v}) \times \text{Monoid}(C^*, \mathbb{M}_{m,v}),$$

so that the ring of functions is the tensor product of the rings of functions of the cyclic factors. Finally, when A^* is cyclic of order n

$$\text{Monoid}(A^*, \mathbb{M}_{m,v}) = \mathbb{M}_{m,v}[n]$$

is the *group* scheme of points of order dividing n, and this is represented by the ring $k[y]/([n](y))$. However, the infinite cyclic group $A^* = \langle z \rangle$ is generated as a monoid by the elements z and z^{-1}, subject to the relation $zz^{-1} = \varepsilon$, so that $\text{Monoid}(A^*, \mathbb{M}_{m,v})$ is represented by the ring $k[y, y']/(y \odot y')$. The case when $k = \mathbb{Z}[v]$ gives the calculation of L_A^m, and hence Theorem 4.1.

Ando suggests the following more general construction, which we hope to investigate further, elsewhere.

Construction B.2 (M. Ando)**.** Given a one-dimensional commutative affine group or formal group $\mathbb{G}$ with a chosen regular coordinate y at the identity, and a group homomorphism $\zeta : A^* \to \mathbb{G}$, we may construct an A-equivariant formal group $\hat{\mathbb{G}}$ by formal completion along the image of ζ. Taking $y(\varepsilon) := y$ to be the coordinate at $\zeta(\varepsilon)$ we obtain an A-equivariant formal group law. We say that such an A-equivariant formal group law is of *type* $(\mathbb{G}, y)$.

Proof. Conditions (Afgl1) and (Afgl2) are automatic, and Condition (Afgl3) follows as in the proof of Proposition 5.1. More precisely, the polynomial ring $k[y]$ is replaced by the ring of functions on $\mathbb{G}$; the hypothesis that y is a regular coordinate means that it is not a zero divisor and that it generates the functions vanishing at $\zeta(\varepsilon)$, so that the argument applies as before. □

Example B.3. (i) A strictly multiplicative A-equivariant formal group law is an A-equivariant formal group law of type $(\mathbb{G}_m, vz)$ for some invertible v. If we allow $\mathbb{G}$ to be the monoid scheme $\mathbb{M}_{m,v}$ we obtain the multiplicative A-equivariant formal group laws with parameter v.

(ii) By the completion theorem of [5], any A-equivariant formal group $\hat{\mathbb{G}}$ law over a ring k complete for the ideal $I = (e(\alpha) \mid \alpha \in A^*)$ is of type $(\mathbb{G}, y)$ where $\mathbb{G}$ is the classical one-dimensional formal group obtained from $\hat{\mathbb{G}}$ by formal completion at the identity. This applies in particular to the A-equivariant formal group laws associated to the cohomology of the Borel construction for a complex oriented cohomology theory [2].

(iii) (M. Ando) Provided A is finite, we may extend the construction to allow $\mathbb{G}$ to be an elliptic curve. Although $\mathbb{G}$ is not affine, we can delete one point of infinite order to obtain an affine scheme and perform Construction B.2.

By construction, if $\mathbb{G}$ is defined over k,

$$A\text{-fgl}_{\mathbb{G},y} = \mathrm{Hom}(A^*, \mathbb{G})$$

as functors on k-algebras, so that the representing ring $L_A^{\mathbb{G},y}$ for A-equivariant formal group laws of type $(\mathbb{G}, y)$ is the ring of functions on $\mathrm{Hom}(A^*, \mathbb{G})$. As before, this is the tensor product of the rings of functions for the cyclic factors of A^*. If A^* is cyclic of order n, we find $\mathrm{Hom}(A^*, \mathbb{G}) = \mathbb{G}[n]$. Since $\mathbb{G}$ is a group rather than a monoid, $\mathrm{Hom}(A^*, \mathbb{G}) = \mathbb{G}$ when A^* is infinite cyclic. As described earlier, $\mathrm{Monoid}(A^*, \mathbb{G})$ is the kernel of the multiplication map of $\mathbb{G}$.

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